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Group divisible designs with block size four and two groups

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This paper is dedicated to Professor Jennifer Seberry

Abstract

We give some constructions of new infinite families of group divisible designs, GDD$(n, 2, 4; \lambda_1, \lambda_2)$, including one which uses the existence of Bhaskar Rao designs. We show the necessary conditions are sufficient for $3 \leqslant n \leqslant 8$. For $n = 10$ there is one missing critical design. If $\lambda_1 > \lambda_2$, then the necessary conditions are sufficient for $n \equiv 4, 5, 8 \pmod{12}$. For each of $n=10$, 15, 16, 17, 18, 19, and 20 we indicate a small minimal set of critical designs which, if they exist, would allow construction of all possible designs for that n. The indices of each of these designs are also among those critical indices for every n in the same congruence class mod 12.

Keywords: Bhaskar Rao design; BIBD; BRD; GDD; Group divisible design; Resolvable designs; Two-associate class PBIBD

1. Introduction

A group divisible design GDD$(n, m, k; \lambda_1, \lambda_2)$ is a collection of k-element subsets of a v-set X called blocks which satisfies the following properties: the $v = nm$ elements of X are partitioned into m subsets (called groups) of size n each; points within the same group are called first associates of each other and each pair of points from the same group appear together in λ_1 blocks; any two points not in the same group are second associates and appear together in λ_2 blocks. Each point of X appears in r of the b blocks, and r is called the replication number.

Designs of the type discussed here have been called both GDDs and group divisible PBIBDs (partially balanced incomplete block designs) and we refer the reader to [6,14] and the references therein. When $\lambda_1 = 0$ and $\lambda_2 = 1$ then exponential notation $k - \mathrm{GDD}(n^m)$ is usually used for GDD with m groups of size n and block size k [12]. In [7] the existence problem was settled for GDDs with first and second associates with block size $k = 3$ and with m groups each of size n with $m, n \geqslant 3$. The problem of necessary and sufficient conditions for $m = 2$ or $n = 2$, and block size three, was established in [6]. Unless otherwise stated, $m = 2$ is assumed from now on.

For GDDs with block size four and two groups, the replication number r and the number of blocks b are given by

$$r = (\lambda_1(n-1) + n\lambda_2)/3,$$

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Table 1
Congruence restrictions for GDDs with 2 groups and block size 4

λ_2	λ_1		
	0 mod 3	1 mod 3	2 mod 3
0 mod 6	Any n	$n \equiv 1 \pmod 3$	$n \equiv 1 \pmod 3$
1 mod 6	$n \equiv 0 \pmod 6$	$n \equiv 2 \pmod 6$	Impossible
2 mod 6	$n \equiv 0 (\text{mod} 3)$	Impossible	$n \equiv 2 \pmod 3$
3 mod 6	n even	$n \equiv 4 \pmod 6$	$n \equiv 4 \pmod 6$
4 mod 6	$n \equiv 0 \pmod 3$	$n \equiv 2 \pmod 3$	Impossible
5 mod 6	$n \equiv 0 \pmod 6$	Impossible	$n \equiv 2 \pmod 6$

and

$$b = n(\lambda_1(n-1) + n\lambda_2)/6 = rn/2.$$

As r and b must be integers, these two necessary conditions on b and r determine possibilities for the parameter n and the indices λ_1 and λ_2 which are summarized in Table 1.

Table 1 does not give the whole story, however. There are several other known necessary conditions [11].

Lemma 1. *Suppose* GDD$(n, 2, 4; \lambda_1, \lambda_2)$ *has replication number r, and suppose b is the number of blocks. Let s, t and u denote the number of blocks with* 4, 3 *and* 2 *points, respectively, from the same group. The necessary conditions for the existence of the GDD include the following*:

(a) $\lambda_1(n-1) + n\lambda_2 \equiv 0 \pmod 3$, *and* $n(\lambda_1(n-1) + n\lambda_2) \equiv 0 \pmod 6$.
(b) $b \geqslant \max\{2r - \lambda_1, 2r - \lambda_2\}$.
(c) $\lambda_2 \leqslant 2\lambda_1(n-1)/n$.
(d) $6s + 3t + 2u = 2\lambda_1 \binom{n}{2} = n(n-1)\lambda_1$, *and* $3t + 4u = n^2\lambda_2$, *and* $s + t + u = b$.

There are two special types of GDD$(n, 2, 4; \lambda_1, \lambda_2)$ called *odd* and *even* which we investigated in [11]. A GDD$(n, 2, 4, \lambda_1, \lambda_2)$ is called odd if each block intersects each of the two groups in exactly one or three points, and it is called even if each block intersects each group in exactly two points.

Lemma 2 (*Hurd and Sarvate [11]*). (a) *For even* GDDs, $n\lambda_2 = 2\lambda_1(n-1)$. *Moreover, if n is even, even* GDDs *exist with* $\lambda_1 = n/2$ *and* $\lambda_2 = n-1$. *For n odd, even* GDDs *exist with* $\lambda_1 = n$ *and* $\lambda_2 = 2(n-1)$. (b) *For any odd* GDD, $\lambda_1 = tn$ *and* $\lambda_2 = t(n-1)$ *for some t. The minimum possible t is $t = 1$ for* $n \equiv 0, 1 \pmod 3$ *and $t = 3$ for* $n \equiv 2 \pmod 3$.

We emphasize that, when $\lambda_1 < \lambda_2$, then necessarily,

$$n/2(n-1) \leqslant \lambda_1/\lambda_2,$$

and Lemma 2 says the inequality is sharp. Further, when there are only two groups, necessarily, $0 < \lambda_1$ and $\lambda_2 < 2\lambda_1$.

A purpose of this note is to present some new designs for small n and to show the necessary conditions are sufficient for their existence. We do this for $3 \leqslant n \leqslant 8$. These results are applied to larger designs for n in the same congruence class mod 12. We indicate several important (critical) designs for each n the existence of which would complete that case. We also exploit some new constructions for block size four GDDs with two groups ($m = 2$). Two of them create several families with large n and small λ_2 (=1, 2, 3). The other uses c-Bhaskar Rao designs (BRDs) (defined in the next section) in the construction resulting in families with large n and with $\lambda_1 < \lambda_2$.

Jennifer Seberry coined BRD in late 1970s and since then there have been many papers on BRDs, c-BRDs, generalized BRDs and their applications, particularly applications in construction of other designs. See [4,5,9] and their references. We are pleased in this note, dedicated to Dr. Seberry, to be able to apply this earlier work here. An interesting aspect of Seberry's work has been her talent in constructing smaller designs by hand. Sometimes such work has then proved useful in leading to general results. For example, examples in [13] led to the wide ranging generalizations in [3]. We are also pleased then in this note to present several new designs found "by hand" and which have intricate combinatorial structure. One of these has already led to an important general construction (Theorem 4).

2. General constructions

In this section we introduce several constructions, give important applications of them, and consider a few special cases. We begin with a set of elementary constructions.

Lemma 3. (a) *Suppose there exists* BIBD$(n, 4, t)$ *and* BIBD$(2n, 4, s)$. *Then there exists* GDD$(n, 2, 4; xt + ys, ys)$ *for any positive integers x and y*. (b) *Suppose there exists a* BIBD$(2n, 4, t)$ *and a* GDD$(n, 2, 4; \lambda_1, \lambda_2)$. *Then there exists a* GDD$(n, 2, 4; tx + \lambda_1, tx + \lambda_2)$. (c) *Suppose there exists* BIBD$(n, 4, t)$ *and* GDD$(n, 2, 4; \lambda_1, \lambda_2)$. *Then there exists a* GDD$(n, 2, 4; xt + \lambda_1, \lambda_2)$.

We note part (a) always results in $\lambda_1 > \lambda_2$. In essence, part (b) of the lemma says that one can add the blocks of a BIBD on the $2n$ points of a GDD and create a new GDD increasing both indices by the same amount. In part (c) one is adding the blocks of a BIBD on each group separately so that only the first index increases. Lemma 1 is used repeatedly in the next section. We next present a general construction for any block size k which was motivated by the GDD(6, 2, 4; 3, 1) in the next section.

Theorem 4. *If there exists* GDD$(n, 2u, k; 0, \lambda_2)$ *and* GDD$(n, u, k; \lambda_1, \lambda_1-\lambda_2)$, *then there exists* GDD$(un, 2, k; \lambda_1, \lambda_2)$. *If there exists* GDD$(n, 2, 4; \lambda_1, \lambda_1-\lambda_2)$ *and* GDD$(n, 4, 4; 0, \lambda_2)$, *in particular, then there exists a* GDD$(2n, 2, 4; \lambda_1, \lambda_2)$.

Proof. Suppose $Y_1 =$ GDD$(n, 2u, k; 0, \lambda_2)$ has groups $A_1, \ldots, A_{2u}$. Then, using $A_1, \ldots, A_u$ as groups, let $Y_2 =$ GDD$(n, u, k; \lambda_1, \lambda_1 - \lambda_2)$. Using $A_{u+1}, \ldots, A_{2u}$ as groups, let $Y_3 =$ GDD$(n, u, k; \lambda_1, \lambda_1 - \lambda_2)$. The blocks of the desired GDD are those of Y_1, Y_2, and Y_3, and the groups are $A_1 \cup \cdots \cup A_u$ and$A_{u+1} \cup \cdots \cup A_{2u}$. □

This is a very useful theorem, especially for construction of GDDs with 2 groups, since GDD$(n, 4, 4; 0, 1)$ exist for all $n \geqslant 3$, except $n = 6$ [12]. The following powerful corollary is especially useful combined with the results in the next section.

Corollary 5. *Suppose a* GDD$(n, 2, 4; \lambda_1, \lambda_2)$ *exists for* $n \geqslant 3$, $n \neq 6$, *and* $\lambda_1 > \lambda_2$. *Then a* GDD$(4^t n, 2, 4; \lambda_1, \lambda_2)$ *exists*.

Proof. When Theorem 4 is applied twice, then the original indices return. □

Theorem 6. *Suppose* $n \geqslant 3$ *and* $n \neq 6$.

(a) *If* $n \equiv 0, 1 (\mathrm{mod}\ 3)$, *then a* GDD$(2n, 2, 4; n, 1)$ *exists*.
(b) *If* $n \equiv 2 (\mathrm{mod}\ 3)$, *then a* GDD$(2n, 2, 4; 3n, 3)$ *exists*.

Proof. This follows immediately from Theorem 4 and Lemma 2. □

Theorem 7. *Suppose* $j \geqslant 1$.

(a) *A* GDD$(12j, 2, 4; 12j, 1)$ *exists*.
(b) *Suppose there exist* $Y_1 =$ BIBD$(3u, 4, 3u\lambda)$ *and* $Y_2 =$ PBD$(3u, \{3, 4\}, \lambda)$ *such that the blocks of size three form a parallel class. Then a* GDD$(3u, 2, 4; 3u\lambda, 1)$ *exists*.
(c) *Suppose there exists* $Y_1 =$ BIBD$(3u, 4, 3u\lambda)$ *and* $Y_2 =$ GDD$(3, u, 4; \lambda - 1, \lambda)$. *Then there exists a* GDD$(3u, 2, 4; 3u\lambda, 1)$.

Proof. For part (a), let $Y =$ GDD$(3, 4j, 4; 0, 1)$, i.e., a 4-GDD(3^{4j}), which is known to exist [10]. Use the blocks of $12j$-copies of Y. For the i-th copy of Y, augment each group with new point x_i to make blocks of size 4. Now use

Table 2
Pairs into blocks

a_i	a_i	a_i	a_i	a_i	a_i	a_j	a_j	a_j
a_j	a_j	a_j	a_k	a_k	a_k	a_k	a_k	a_k
b_i	b_i	b_j	b_i	b_i	b_j	b_i	b_i	b_j
b_j	b_k	b_k	b_j	b_k	b_k	b_j	b_k	b_k

the blocks of a copy of BIBD$(12j, 4, 12j)$ based on points $\{x_1, x_2, \ldots, x_{12j}\}$. This creates a GDD$(12j, 2, 4; 12j, 1)$ where the points of Y give one group and the points of the set $\{x_1, x_2, \ldots, x_{12j}\}$ give the second group. For part (b), take $n = 3u$ copies of Y_2 based on points $\{1, 2, \ldots, n\}$. These copies will have n parallel classes of triples. Augment the triples in the i-th parallel class with new element x_i for $1 \leqslant i \leqslant n$. Use the blocks of a BIBD$(n, 4, n\lambda)$ based on points $\{x_1, \ldots, x_n\}$. For part (c), use the groups of Y_2 as a parallel class and apply (b). $\square$

Example 8. The blocks $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 4, 6\}$, $\{1, 2, 5, 6\}$, $\{1, 3, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{1, 3, 5, 6\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 4, 6\}$, $\{2, 3, 5, 6\}$ give a PBD$(6, \{3, 4\}, 4)$ in which the blocks of size 3 partition the set of points. By the previous theorem, a GDD$(6, 2, 4; 24, 1)$ exists. Of course, the blocks of size 4 give a GDD$(3, 2, 4; 3, 4)$ design, illustrating that the parts of the theorem are variations on the same theme.

In the rest of this section we describe and exploit a new construction which gives us several families of GDDs. The construction is based on the existence of BRDs. A BRD(v, k, λ), is identified with a $\{0, 1, -1\}$-matrix with the property that each pair of rows has inner product 0 and satisfying the condition that when the -1's are changed to 1, the matrix becomes the incidence matrix of a BIBD(v, k, λ). BRD$(v, 4, 4u)$s are known to exist [5,8,9] whenever the underlying BIBD exists.

Theorem 9. *Suppose there exist a* BIBD$(n, 3, \lambda)$ *and a* BRD$(n, 4, 4\lambda(n-3))$. *Then there exists a* GDD$(n, 2, 4; \lambda_1, \lambda_2)$ *where* $\lambda_1 = \lambda(2n - 3)$, *and* $\lambda_2 = \lambda(2n - 2)$.

Proof. Let X_1 and X_2 be copies of the same BIBD$(n, 3, \lambda)$ based on the points of $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, respectively. The sets A and B will be the two groups of the GDD. Using X_1 and X_2 first, from the two corresponding blocks $\{a_i, a_j, a_k\}$ and $\{b_i, b_j, b_k\}$ we construct 9 even blocks with 4 points each as shown in Table 2.

This gives $9\lambda n(n-1)/6$ even blocks. Observe that the pair $\{a_i, b_i\}$ has occurred $4\lambda(n-1)/2=2\lambda(n-1)$ times in these blocks, for $i=1, \ldots, n$, but the pair $\{a_i, b_j\}$, $i \neq j$, has occurred 4λ times. The pairs $\{a_i, a_j\}$ and $\{b_i, b_j\}$ have occurred 3λ times. Now we construct the other blocks. Suppose $Y = \text{BRD}(n, 4, \beta)$ exists with $\beta = 2(2\lambda(n-1) - 4\lambda) = 4\lambda(n-3)$. It has replication number $r = 4\lambda(n-1)(n-3)/3$ and b blocks with $b = \lambda n(n-1)(n-3)/3$. Take two copies of Y, based on the points of A and B. In the first replace any occurrence of $-a_i$ in a signed block with b_i and in the second replace any occurrence of $-b_i$ in a signed block with a_i $(i = 1, \ldots, n)$. We have created $2\lambda n(n-1)(n-3)/3$ new blocks in which the pairs $\{a_i, b_i\}$ do not appear together but every other pair of points from either X_1 or X_2 meet γ times, where $\gamma = 2\lambda(n-3)$. The two collections of blocks taken together give a GDD$(n, 2, 4; \lambda_1, \lambda_2)$ where

$$\lambda_1 = 2\lambda(n-3) + 3\lambda = \lambda(2n-3),$$
$$\lambda_2 = 2\lambda(n-3) + 4\lambda = \lambda(2n-2). \quad \square$$

Example 10. To construct a GDD$(7, 2, 4; 11, 12)$, use a BIBD$(7, 3, 1)$ and a BRD$(7, 4, 16)$. To construct a GDD$(9, 2, 4; 15, 16)$, use a BIBD$(9, 3, 1)$ and a BRD$(9, 4, 24)$.

Corollary 11. (a) *If* $n \equiv 1, 3 \pmod 6$, *then a* GDD$(n, 2, 4; 2n-3, 2n-2)$ *exists*. (b) *If* $n \equiv 0, 1 \pmod 3$, *then a* GDD$(n, 2, 4; 2(2n-3), 2(2n-2))$ *exists*. (c) *If* $n \equiv 5 \pmod 6$, *then a* GDD$(n, 2, 4; 3(2n-3), 3(2n-2))$ *exists*.

Proof. Apply Theorem 8 since, for these parameters, BIBD$(n, 3, \lambda)$ exist for $\lambda = 1, 2, 3$ in parts (a)–(c), respectively, and the BRD$(n, 4, 4(n-3))$ exist as well. $\square$

In general we note that these parameters deal with cases not immediately available from the even designs.

3. GDDs with small group sizes

We comment that when the parameters allow for extensive application of Lemma 3, then many results are quickly obtained ($n=8$) and similarly, if we construct the designs from the odd and even designs and use Lemma 3, we obtain the needed GDDs ($n=3, 4, 5$) but otherwise, results tend to get harder to obtain because the necessary conditions allow many rich possibilities ($n=6, 7$). We will list for each of $n = 10$, 15, 16, 17, 18, 19, and 20 a minimal set of “critical” indices, and the existence of designs with these indices would allow construction of all remaining designs for that n. It is striking that the minimal even design indices for a particular n automatically become a part of a critical set of indices for $n+12$. As will be seen, the difficulty is cumulative: the critical design indices for $n+12j$ include the even design parameters for $n+12k$ for $1 \leqslant k < j$.

The case $n = 3$: For this case the necessary conditions (Table 1 and Lemma 1) are that λ_1 is necessarily a multiple of three, that λ_2 is even, and that $3\lambda_2 \leqslant 4\lambda_1$. When $n = 3$ (and only for this case) part (b) of Lemma 1 is relevant as well.

Lemma 12. *A necessary condition for a* GDD$(3, 2, 4; \lambda_1, \lambda_2)$ *is that* $3\lambda_2 \leqslant 4\lambda_1 \leqslant 6\lambda_2$.

Proof. When $n = 3$, $r = 2\lambda_1/3 + \lambda_2$ and $b = \lambda_1 + 3\lambda_2/2$. Since $b \geqslant 2r - \lambda_2$ (Lemma 1(b)) we have, on substitution, that $3\lambda_2 \geqslant 2\lambda_1$. The full inequality follows from part (c) of Lemma 1. □

By way of contrast, when $n \geqslant 4$, part (c) of Lemma 3 shows λ_1 is not bounded above by any multiple of λ_2.

If $n = 3$, an even GDD$(3, 2, 4; 3, 4)$ exists and an odd GDD$(3, 2, 4; 3, 2)$ exists (Lemma 2). Multiples of these are the extreme cases in the previous inequality.

Lemma 13. *A* GDD$(3, 2, 4; 3x, 2y)$ *exists for all* x, y *such that* $y \leqslant 2x \leqslant 2y$.

Proof. To get a GDD$(3, 2, 4; 3x, 2y)$ use $y-x$ copies of a GDD$(3, 2, 4; 3, 4)$ and $2x-y$ copies of GDD$(3, 2, 4; 3, 2)$. □

Theorem 14. *The necessary conditions are sufficient for the existence of* GDD$(3, 2, 4; \lambda_1, \lambda_2)$.

Suppose more generally, that $n \equiv 3 \pmod{12}$, $n \geqslant 15$. Then as above, $\lambda_1 \equiv 0 \pmod 3$ and λ_2 is even. The other necessary condition is that $(3 + 12j)/(4 + 24j) \leqslant \lambda_1/\lambda_2$. By an application of Lemma 3, and using the odd and even designs, we can say the following:

Theorem 15. *The designs* GDD$(3+12j, 2, 4; \lambda_1, \lambda_2)$ *exist with* (λ_1, λ_2) *given by* $\lambda_1 = (3+12j)s + (3+12j)t + 6u + 6v$, *and* $\lambda_2 = (4 + 24j)s + (2 + 12j)t + 6v$.

For each n in this class, we need the critical designs GDD$(n, 2, 4; 3, 2)$ and GDD$(n, 2, 4; 3, 4)$. In general, as n grows (within the class) more designs are needed to complete the case. For example, construction for the next case in this congruence class, $n = 15$, would require the designs for (λ_1, λ_2)-pairs of (3, 2), (3, 4) and (6, 2). Then repeated applications of Lemma 3 would give all designs with $\lambda_1 > \lambda_2$. The case for $n = 15$ also requires using the designs for (λ_1, λ_2)-pairs of (3, 4), (6, 10), (9, 16), (12, 22) and (15, 28). But only (15, 28), the even design, is known. Moreover, for $n \equiv 3 \pmod{12}$ and $n > 15$, the (15, 28) design becomes a critical one.

The case $n = 4$: Table 1 tells us in this case λ_2 must be a multiple of 3 and there are no restrictions on λ_1 in the table. The even GDD$(4, 2, 4; 2; 3)$ is the design of critical index since by Lemma 1(c), when $n = 4$, $2\lambda_2 \leqslant 3\lambda_1$. As each group may form one block, we can increase the first index by any amount. Hence a GDD$(4, 2, 4; u + 2x, 3x)$ exists for any $u \geqslant 1$ using x copies of the even design. The same construction applies to $n \equiv 4 \pmod{12}$ as well since BIBD$(4 + 12t, 4, 1)$ exist.

Theorem 16. *The necessary conditions are sufficient for a* GDD$(4, 2, 4; \lambda_1, \lambda_2)$.

Theorem 17. *Suppose* $n = 4 + 12j$ *for* $j > 0$.

(a) *The existence of a* GDD$(n, 2, 4; u, v)$ *implies the existence of a* GDD$(n, 2, 4; u + 3s + w, v + 3s)$ *for any* w, s.
(b) GDD$(4 + 12j, 2, 4; (2 + 6j)t, (3 + 12j)t)$ *and* GDD$(4 + 12j, 2, 4; (2 + 6j)t, (3 + 12j)t)$ *exist for all* $t > 0$.
(c) *The necessary conditions are sufficient for a* GDD$(4 + 12j, 2, 4; \lambda_1, \lambda_2)$, *if* $\lambda_1 > \lambda_2$.

A critical sequence of designs here is GDD$(4 + 12j, 2, 3; 2, 3)$. Its solution for $n = 4 + 12j$ is necessary for the case for that n. As an example, when $n = 16$ there exists GDD(16, 2, 4; 8, 15) and hence GDD(16, 2, 4; λ_1, 15) for all $\lambda_1 \geqslant 8$. Similarly, there exists GDD(16, 2, 4; λ_1, 30) for $\lambda_1 \geqslant 16$. However, there is (as yet) no GDD(16, 2, 4; 15, 27). A solution for the two (λ_1, λ_2) pairs (2, 3) and (5, 9) would solve the existence problem for $n = 16$.

The case $n = 5$: We first observe that $\lambda_2 \equiv 0 (\mathrm{mod}\, 2)$, by Table 1, and that necessarily $\lambda_1 \equiv \lambda_2 (\mathrm{mod}\, 3)$.

The existence of GDD(5, 2, 4; 5, 2) is possible according to Table 1, and this is the GDD with the smallest possible indices for $n = 5$. A direct construction is to use one copy of a BIBD(10, 4, 2) and the blocks of one copy of BIBD(5, 4, 3) on each group. By Lemma 3, we get the family $F = \{\mathrm{GDD}(5, 2, 4; 2x + 3y, 2x) : x, y \geqslant 1\}$. Another small design is GDD(5, 2, 4; 7, 4), a member of family F. Indeed, the conditions so far show that, if $\lambda_1 > \lambda_2$, then a GDD(5, 2, 4; λ_1, λ_2,) has the parameters of a member of family F. GDD(5, 2, 4; 5, 8), the even design, is not in family F, and here $\lambda_2 > \lambda_1$.

We claim that if $\frac{5}{8} \leqslant \lambda_1/\lambda_2$ (Lemma 1(c)) and if $\lambda_1 \equiv \lambda_2 (\mathrm{mod}\, 3)$, and if $\lambda_1 < \lambda_2$, then there is a X=GDD(5, 2, 4; λ_1, λ_2). This is most easily seen by setting $\lambda_2 = \lambda_1 + 3j$. We will use j copies of an even GDD(5, 2, 4; 5, 8) and $(\lambda_1 - 5j)/2$ copies of a BIBD(10, 4, 2). The first index is

$$5j + 2(\lambda_1 - 5j)/2 = 5j + \lambda_1 - 5j = \lambda_1.$$

The second index is

$$8j + 2(\lambda_1 - 5j)/2 = 8j + \lambda_1 - 5j = \lambda_1 + 3j = \lambda_2.$$

Since the second index is even, $\lambda_1 + 3j = 2t$ for some t. Thus, $\lambda_1 = 2t - 3j$ and $\lambda_1 - 5j = 2t - 8j$ which is even. Note that, $\lambda_1 - 5j \geqslant 0$. To see this observe $j = (\lambda_2 - \lambda_1)/3$ and

$$\lambda_1 - 5j = \lambda_1 - 5(\lambda_2 - \lambda_1)/3 = (8\lambda_1 - 5\lambda_2)/3.$$

But, $8\lambda_1 - 5\lambda_2 \geqslant 0$ by Lemma 1(c). Further, if $\lambda_1 - 5j = 0$, then the desired GDD is a multiple of the even design. This proves the following theorem.

Theorem 18. *The necessary conditions are sufficient for existence of a* GDD $(5, 2, 4; \lambda_1, \lambda_2)$.

Theorem 19. *The family of designs* GDD$(5 + 12j, 2, 4; 2t + 3s, 2t) : s, t \geqslant 1,\ j \geqslant 0$ *all exist*, *that is*, *the necessary conditions are sufficient for* $n \equiv 5 (\mathrm{mod}\, 12)$ *provided* $\lambda_1 > \lambda_2$.

Although BIBD$(5+12j, 4, 3)$ and BIBD$(10+24j, 4, 2)$ exist, the method used here for $n=5$ does not (yet) extend in full to $n = 17$ (or $n \equiv 5 (\mathrm{mod}\, 12)$) because, as earlier, the even design indices are too large. One example: the design GDD(17, 2, 4; 39, 72) should exist but is not constructible using the "nearby" even design GDD(17, 2, 4; 34, 64). The needed critical designs for $n = 17$ are the (λ_1, λ_2) pairs: (5, 8), (8, 14), (11, 20) and (14, 26).

The case $n = 6$: In this case, λ_1 must be a multiple of 3, but λ_2 may be any number, using Table 1. We observe that the pairs of possible indices for $n = 6$ include (3, 1), (3, 2), (3, 3), (3, 4), and (3, 5), but (3, 6) is not possible since necessarily $3/5 \leqslant \lambda_1/\lambda_2$ (Lemma 1(c)). GDD(6, 2, 4; 3, 1) exists and blocks are the columns in Table 3. In this table and the examples that follow the numbers represent one group and the letters represent the second group.

The solution arrived at by hand led us to the very useful Theorem 4. The 4-by-9 segment is $Y_1 = \mathrm{GDD}(3, 4, 4; 0, 1)$ with groups $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5, 6\}$, $A_3 = \{a, b, c\}$, and $A_4 = \{d, e, f\}$. The 4-by-12 segment shows Y_3 and Y_2 together.

Example 20. We construct a GDD(6, 2, 4; 6, 1) whose blocks are in Table 4.

Lemma 21. *The necessary conditions are sufficient for the existence of* GDD(6, 2, 4; λ_1, 1).

Table 3
Blocks for GDD(6, 2, 4; 3, 1)

1	1	1	4	4	4	a	a	a	d	d	d
2	2	2	5	5	5	b	b	b	e	e	e
3	3	3	6	6	6	c	c	c	f	f	f
4	5	6	1	2	3	d	e	f	a	b	c

a	b	c	b	c	a	c	a	b
d	e	f	f	d	e	e	f	d
1	1	1	2	2	2	3	3	3
4	5	6	4	5	6	4	5	6

Table 4
A GDD(6, 2, 4; 6, 1)

1	2	3	4	5	6	a	a	b	a	d	b
2	4	4	1	3	1	b	c	c	d	e	c
3	5	6	5	6	2	e	e	f	f	f	d
b	a	f	e	d	c	6	3	5	1	2	4

1	1	1	1	1	1	1	1	2	2	2	2
2	2	2	2	3	3	3	4	3	3	3	4
3	3	4	5	4	4	5	5	4	4	5	5
4	5	6	6	5	6	6	6	5	6	6	6

a	a	a	a	a	a	a	a	b	b	b	c
b	b	b	b	b	c	c	c	d	c	c	d
d	e	d	c	c	d	e	d	e	e	d	e
e	f	f	f	d	e	f	f	f	f	e	f

Table 5
A GDD(6, 2, 4; 3, 2)

a	b	c	d	e	f	1	2	3	4	5	6
1	2	1	1	2	2	b	a	b	b	a	a
3	5	3	4	4	3	e	c	d	c	c	d
4	6	6	5	5	6	f	d	e	f	f	e

1	1	3	a	a	c
2	2	4	b	b	d
3	5	5	c	e	e
4	6	6	d	f	f

1	3	4	3	4	2	1	1	2
2	5	6	5	6	4	6	5	3
a	a	a	b	b	c	c	d	d
b	e	f	c	d	e	e	f	f

Proof. Since the designs exist for $\lambda_1 = 3, 6$, the result follows by Lemma 3. $\square$

We construct a GDD(6, 2, 4; 3, 2), for which $r = 9$ and $b = 27$ in Table 5.

We exhibit (part of) a GDD(6, 2, 4; 3, 4) in Table 6. Our solution was to use the matrix of 9 blocks and the matrix of 12 blocks from Table 5. Eighteen more blocks were necessary, and we replaced the six full blocks in Table 5 which have 4 points from the same group with the corresponding 18 pairs from each group. We used these two sets of 18 pairs to make 18 blocks with 2 points from each group. The new blocks are shown in Table 6.

As an aside, we would like to mention a curious array which we put in Table 7. In constructing the new blocks, four of the remaining pairs occurred twice, and handling them was inconvenient. Ultimately, we put several pairs twice each

Table 6
New blocks for GDD(6, 2, 4; 3, 4)

1	1	1	2	3	3
2	2	3	4	4	4
a	c	e	e	a	c
b	d	f	f	b	d

1	1	1	2	2	2	3	3	4	4	5	5
4	5	6	3	5	6	5	6	5	6	6	6
a	d	b	b	c	a	a	c	b	d	b	a
c	e	f	d	f	e	f	e	e	f	c	d

Table 7
An array of available pairs from the same group

$\overline{ac}$	AE	af	ad
bf	bc	BD	$\overline{be}$
de	$\overline{df}$	ce	CF

into the first matrix of blocks. In the remaining set of blocks we had to pair each of {1, 2, 3, 4} with each of the six letters exactly once and {5, 6} with each exactly twice—only the pair {5, 6} still needed to occur twice. Our method for the last 4-*by*-12 segment was first to array the remaining pairs of letters into Table 7, and the partitions of three pairs into groups appeared eight times in useful combinations.

It will be convenient to refer to the following observation:

Theorem 22. *If a* GDD$(6 + 12j, 2, 4; \lambda_1, \lambda_2)$ *exists, then so does a* GDD$(6 + 12j, 2, 4; \lambda_1 + 6x + 3y, \lambda_2 + 3y)$ *for any nonnegative integers x and y.*

Proof. Apply the fact that BIBD$(6 + 12j, 4, 6)$ and BIBD$(12m, 4, 3)$ exist and use Lemma 3. □

We note that an even GDD(6, 2, 4; 3, 5) exists, and an odd GDD(6, 2, 4; 6, 5) exists. We have now constructed all the designs for $n = 6$ and $\lambda_1 = 3$. We will use these to complete the case for $n = 6$ and $\lambda_1 = 6$.

When $n=6$ and $\lambda_1=6$, λ_2 is bounded by $1 \leqslant \lambda_2 \leqslant 10$. We have constructed the GDD$(6, 2, 4; \lambda_1, \lambda_2)$ with indices (6, 1) and (6, 5). Referring to GDDs with just the indices, a (6, 2) is available using two copies of a (3, 1). A (6, 3) is obtained from a (3, 1) and (3, 2). For a (6, 4), (6, 8) and (6, 10), use two copies of a (3, 2), (3, 4) and (3, 5), respectively. A (6, 7) is obtained from a (3, 4) and (3, 3) (a BIBD). A (6, 9) is obtained from a (3, 4) and a (3, 5). A GDD$(6, 2, 4; 3j, 1)$ is obtained from a (3, 1) or (6, 1) and enough copies of a BIBD(6, 4, 6) on each group. Arguing inductively, suppose we have constructed all GDD$(6, 2, 4; \lambda_1, \lambda_2)$ up to and including GDD$(6, 2, 4; 3(j-1), 5(j-1))$ for $j \geqslant 2$. When $\lambda_2 > 1$, a GDD$(6, 2, 4; 3j, \lambda_2)$ is obtained from a GDD$(6, 2, 4; 3(j-1), \lambda_2 - x)$ and a suitable $(3, x)$ with $1 \leqslant x \leqslant 5$. We now have:

Theorem 23. *The necessary conditions are sufficient for the existence of a* GDD$(6, 2, 4; \lambda_1, \lambda_2)$.

Proof. For any pair (u, v) of indices, if $u < v$, use the preceding argument. Otherwise, apply Theorem 22. □

For $n = 18$, the critical designs are (3, 1), (3, 2), (3, 4), (3, 5), (6, 1), and (6, 11).

The case $n = 7$: In this case, λ_2 must be a multiple of six, but there are no other restrictions from Table 1 on λ_1. The even design GDD((7, 2, 4; 7, 12) and the odd design GDD(7, 2, 4; 7, 6) are known, but Table 1 and Lemma 1(c) allow $(\lambda_1, \lambda_2) = (4, 6)$ and $(\lambda_1, \lambda_2) = (5, 6)$. Both these designs exist, and we outline an argument to show this existence.

Table 8
The odd blocks for GDD(7, 2, 4; 4, 6)

1	2	3	4	5	6	7	a	b	c	d	e	f	g
2	3	4	5	6	7	1	b	c	d	e	f	g	a
4	5	6	7	1	2	3	d	e	f	g	a	b	c
g	a	b	c	d	e	f	7	1	2	3	4	5	6

Table 9
The even blocks for a GDD(7, 2, 4; 4, 6)

a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	b	b	c	c	c	d	d	d	e	e	e	f	f	f	g	g	g
1	3	5	1	2	4	1	3	6	1	2	3	1	2	4	1	2	4
2	6	7	3	6	5	4	5	7	5	7	4	6	3	7	7	5	6

b	b	b	b	b	b	b	b	b	b	b	b	b	b	b	c	c	c
c	c	c	d	d	d	e	e	e	f	f	f	g	g	g	d	d	d
2	4	6	2	5	3	2	4	7	2	4	1	2	3	5	3	1	7
3	7	1	4	6	7	5	6	1	6	5	3	7	4	1	4	5	1

c	c	c	c	c	c	c	c	c	d	d	d	d	d	d	d	d	d
e	e	e	f	f	f	g	g	g	e	e	e	f	f	f	g	g	g
3	1	6	3	1	5	3	5	2	4	6	1	4	2	7	4	6	2
5	4	7	6	2	7	7	5	4	5	2	3	6	5	1	7	1	3

e	e	e	e	e	e	f	f	f
f	f	f	g	g	g	g	g	g
5	7	1	5	1	3	6	3	1
6	3	4	7	2	6	7	5	4

Table 10
Some new blocks for a GDD(7, 2, 4; 5, 6)

1	1	1	2	2	2		a	a	a	b	b	b
2	2	2	3	3	3		b	b	b	c	c	c
4	4	4	5	5	5	...	d	d	d	e	e	e
a	b	d	b	c	e		1	2	4	2	3	5

For GDD(7, 2, 4; 4, 6) we have $r = 22$, $b = 77$. The odd blocks we need are in Table 8. As usual, the numbers represent one group and the letters represent the other group.

A scheme for constructing even blocks based on those in Table 8 led to the blocks in Table 9, and the two tables together show the existence of a GDD(7, 2, 4; 4, 6).

For a GDD(7, 2, 4; 5, 6), $r = 24$, $b = 84$. The idea of using the same odd blocks as in the previous case three times each naturally occurred. Our solution, however modifies this idea. We use each of the blocks of a BIBD(7, 3, 1) three times, but now each is augmented with a corresponding point. Several of these blocks are shown in Table 10.

The letters a, b, and d correspond to the numbers in the first block {1, 2, 4} of the triple system. In these blocks, the pair {2, a} appears twice, once in a block with 1 and 4, and once in a corresponding block with a and d. They need to appear four more times in some block together. This leads to the remainder of the solution here: discard the first block in each of the 21 triads in Table 9 (leaving 42 even blocks, exactly the number needed), and use here the remaining even blocks from the previous design.

Lemma 24. *A* GDD$(7, 2, 4; \lambda_1, 6)$ *exists for all* $\lambda_1 \geqslant 4$.

Proof. Use Lemma 3 and the existence of GDD(7, 4, 2; 4, 6), GDD(7, 4, 2; 5, 6) and a BIBD(7, 4, 2). □

To complete the $n = 7$ case, we would like to apply the argument in the previous lemma for every possible λ_2. First, if $\lambda_2 = 12$, then a $(7, 12)$ exists (the even design), and by using two copies of a $(4, 6)$ we get an $(8, 12)$. Thus, a GDD$(7, 2, 4; \lambda_1, 12)$ exists for $\lambda_1 \geqslant 7$ by the argument in the lemma. In fact, for $\lambda_2 = 12t$ we can say a GDD$(7, 2, 4; \lambda_1, 12t)$ exists for all $\lambda_1 \geqslant 7t$. The argument is the same since a $(7t, 12t)$ exists and from a $(7(t-1), 12(t-1))$ and an $(8, 12)$ we can get a $(7t + 1, 12t)$. It only remains to consider $\lambda_2 = 6 + 12t$. First, observe that $\lambda_1 = 4 + 7t$ is the smallest possible λ_1. To see this

$$\lambda_1/\lambda_2 = (3 + 7t)/(6 + 12t) < (3.5 + 7t)/(6 + 12t) = \tfrac{7}{12}.$$

This inequality is opposite to the requirement in Lemma 1(c). We can obtain $\lambda_1 = 4 + 7t$ and $5 + 7t$ by adding a $(4, 6)$ to the $(7t, 12t)$ and $(7t + 1, 12t)$ designs. Now the argument in the lemma applies here too. This proves:

Theorem 25. *The necessary conditions are sufficient for the existence of* GDD$(7, 2, 4; \lambda_1, \lambda_2)$.

We can apply Lemma 3 for $n = 7$ and again for $n = 14$ as follows.

Theorem 26. *A* GDD$(7 + 12j, 2, 4; \lambda_1, \lambda_2)$ *imply the existence of* GDD$(7 + 12j, 2, 4; \lambda_1 + 2x + 6y, \lambda_2 + 6y)$ *for any nonnegative integers x and y*.

For $j \geqslant 1$, the critical designs here include GDD$(7+12j, 2, 4; \lambda_1, \lambda_2)$ with (λ_1, λ_2) given by $(4, 6)$, $(5, 6)$ and $(7, 12)$. Additionally, for $n = 19$, the remaining critical designs are $(10, 18)$, $(13, 24)$, and $(16, 30)$.

The case $n = 8$: In this case, according to Table 1, $\lambda_1 \equiv \lambda_2 (\mathrm{mod}\ 3)$ and according to Lemma 1(c), $\frac{4}{7} \leqslant \lambda_1/\lambda_2$. We note that BIBD(8, 4, 3) and BIBD(16, 4, 1) exist. By Lemma 3 we have the following:

Theorem 27. (a) *If a* GDD$(8+12j, 2, 4; \lambda_1, \lambda_2)$ *exists, then so does a* GDD$(8+12j, 2, 4; \alpha, \beta)$ *with* $\alpha = x\lambda_1 + 3y + u$ *and* $\beta = x\lambda_2 + u$ *for any* x, y, u. *In particular, a* GDD$(8+12j, 2, 4; 3y+1, 1)$ *exists for all* $y > 0$. (b) *A* GDD$(8, 2, 4; \lambda_1, \lambda_2)$ *exists for* $\lambda_1 = 4t + 3s + u$ *and* $\lambda_2 = 7t + u$ *for any* $t \geqslant 1$ *and* $s, u \geqslant 0$.

Proof. Part (a) follows for the initial remarks which apply for general n. For part (b), use t-copies of the even design GDD(8, 2, 4; 4, 7) and use part (a). □

If $n = 8 + 12j$, change $(4t, 7t)$ in the theorem to $(t(4 + 6j), t(7 + 12j))$, and the theorem applies to n.

Theorem 28. (a) *The necessary conditions are sufficient for* GDD$(8, 2, 4; \lambda_1, \lambda_2)$. (b) *The necessary conditions are sufficient for* GDD$(8 + 12j, 2, 4; \lambda_1, \lambda_2)$ *whenever* $\lambda_1 > \lambda_2$.

Proof. Part (b) follows from the remarks preceding the theorem. For part (a), the necessary conditions on indices mean $\lambda_1 = 4t + x$ and $\lambda_2 = 7t + x$. Since by Lemma 1(c), $\frac{4}{7} \leqslant \lambda_1/\lambda_2$, we have the GDD$(8, 2, 4; \lambda_1, \lambda_2)$ can be formed using t copies of the even design GDD(8, 2, 4; 4, 7) and x copies of a BIBD(16, 4, 1). □

One critical design sequence for $n = 8 + 12j$ is $X_n =$ GDD$(n, 2, 4; 4, 7)$. For $n = 20$, $(4, 7)$ and $(7, 13)$ are critical designs.

We would like to end this paper with some similar results for the case when $n = 10$. In this case, $\lambda_2 \equiv 0 (\mathrm{mod}\ 3)$ but λ_1 may be any number according to Table 1.

Theorem 29. *A* GDD$(10 + 12j, 2, 4; (5 + 6j)t + 2u + 3v, (9 + 12j)t + 3v)$ *exists for all* $j, t, u, v \geqslant 0$.

Proof. Use the even design and copies of BIBD(10, 4, 2) on each group and BIBD(20, 4, 3) on the union of both groups. □

Note that the first critical design for $n = 10 + 12j$ is $X_n = \mathrm{GDD}(n, 2, 4; 2, 3)$ whose existence would immediately give us many missing designs including all those (small) missing designs for $n = 10$.

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